BOGOMOLOV MULTIPLIERS OF SOME GROUPS OF ORDER p^6

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Abstract. Let $p > 3$ be a prime and G be a nonabelian p-group. Let $B_0(G)$ be the Bogomolov multiplier of G. In this paper, we classify all the groups of order p^6 of class ≤ 3 with $B_0(G) \neq 0$.

1. Introduction

Let k be a field, G a finite group and V a faithful finite-dimensional representation of G over k. Let $k(V)$ be the rational function field on which G naturally acts. We write $k(V)^G$ for the corresponding invariant field. The famous Noether's problem asks whether $k(V)^G$ is purely transcendental over k. This problem has close connection with Lüroth's problem and the inverse Galois problem [14, 23, 24, 25]. Consider the pair $(k, G) = (\mathbf{Q}, C_n)$ with the cyclic action on $\mathbf{Q}(V) = \mathbf{Q}(x_1, \cdots, x_n)$, where $\mathbf{Q}$ is the field of rational numbers and C_n is the cyclic group of order n. In 1969, Swan [26] showed that the invariant field $\mathbf{Q}(V)^{C_n}$ is not purely transcendental over $\mathbf{Q}$ when $n = 47, 113, 233$. This is the first counterexample to Noether's problem. But it seems that Swan's method doesn't work on the case of an algebraically closed field, such as the filed of complex numbers $\mathbf{C}$. In 1984, Saltman [25] used the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$ as an obstruction to prove that there exists a p-group of order p^9, G, such that $\mathbf{C}(V)^G$ is not purely transcendental over $\mathbf{C}$. In 1988, Bogomolov [3] proved that the unramified cohomology group $H^2_{\mathrm{nr}}(\mathbf{C}(V)^G, \mathbf{Q}/\mathbf{Z})$ is isomorphic to

$$B_0(G) = \bigcap_{A \in \mathscr{B}_G} \mathrm{Ker}\left\{\mathrm{res}^G_A : H^2(G, \mathbf{Q}/\mathbf{Z}) \to H^2(A, \mathbf{Q}/\mathbf{Z})\right\},$$

where $\mathscr{B}_G$ denotes the set of bicyclic subgroups of G and res^G_A is the usual cohomological restriction map. The group $B_0(G)$ is a subgroup of the Schur multiplier $H^2(G, \mathbf{Q}/\mathbf{Z})$ of G, so $B_0(G)$ is also called the *Bogomolov multiplier* of G ([17]). Bogomolov [3] used the above description to find new examples of groups G of order p^6 with $B_0(G) \neq 0$.

On the other hand, we remark the following result on p-groups of small order.

Theorem 1.1 ([10])**.** *Let p be a prime and G a p-group of order $\leq p^4$. Assume that k is a field of chark $\neq p$ and contains a primitive p^eth root of unity, where p^e is the exponent of G. Then $k(V)^G$ is purely transcendental over k for any linear representation V. In particular, $B_0(G) = 0$.*

Date: April 18, 2013.

2010 *Mathematics Subject Classification.* 13A50, 14E08, 12F12.

Key words and phrases. Noether's problem; Bogomolov multiplier; unramified Brauer group; finite p-group.

A natural problem is to classify the groups of order p^5 and p^6 with nontrivial B_0. But computing the Bogomolov multiplier of a p-group G is a complicated thing. For $p = 2$, a result due to Chu, Hu, Kang and Prokhorov [9] shows that if G is a group of order 32, then $B_0(G)$ is trivial. Later, Chu, Hu, Kang and Kunyavskii [8] classify all the groups G of order 64 with nontrivial B_0. Meanwhile, we notice that for $p \geq 3$, a complete list of all groups of order p^5 and p^6 is well-known by James's work [13], in which the nonabelian groups of order p^5 and p^6 are divided into 9 isoclinism families $\{\Phi_2, \cdots, \Phi_{10}\}$ and 42 isoclinism families $\{\Phi_2, \cdots, \Phi_{43}\}$ respectively.

Recently, Moravec [20] used a notion of the *nonabelian exterior square* $G \wedge G$ of a given group G to obtain a new description of $B_0(G)$. This is an important result. As an application, it is proved in [20] that there are precisely three groups of order 3^5 with nontrivial B_0. Recently, Hoshi, Kang and Kunyavskii proved the following result.

Theorem 1.2 ([12]). *Let $p > 3$ be a prime and G a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the family Φ_{10}.*

We notice that Theorem 1.2 is also proved by purely combinatorial methods in Moravec [21]. In [12], an interesting question asks whether $B_0(G_1)$ is isomorphic to $B_0(G_2)$ for two isoclinic p-groups G_1 and G_2. Moravec answered this question affirmatively.

Theorem 1.3 ([22]). *Let G_1 and G_2 be isoclinic p-groups. Then $B_0(G_1)$ is isomorphic to $B_0(G_2)$.*

This helpful fact means that if we want to discuss the vanishing of B_0 for the groups in some isoclinic family Φ_i, it suffices to pick up one suitable representative $G \in \Phi_i$ and compute the $B_0(G)$. On the other hand, there are also some papers addressing on the vanishing question of the Bogomolov multiplier of finite simple groups, such as [4], [6], [17], and [5].

The purpose of this paper is to compute the Bogomolov multiplier of some groups of order $p^6 (p > 3)$ with class ≤ 3. We will follow the route of Moravec [21] and extend some methods of Hoshi-Kang [11]. In what follows, we always assume that G is a group of order p^6 of class ≤ 3. It follows from the classification of James [13] that G belongs to one of the isoclinism families: $\Phi_2, \cdots, \Phi_8$, $\Phi_{11}, \cdots, \Phi_{22}$, and $\Phi_{31}, \cdots, \Phi_{34}$. The following is our main result.

Theorem 1.4. *Let $p > 3$ be a prime and G a group of order p^6 of class ≤ 3. Then $B_0(G) \neq 0$ if and only if G belongs to one of Φ_{18}, Φ_{20}, and Φ_{21}.*

Remark 1.5. The result that $B_0(G) = 0$ for $G \in \Phi_2, \Phi_8$ or Φ_{14} was also proved recently in Chen [7] and Michailov [19]. Actually, Noether's problem for these groups has an affirmative answer if the ground field contains a primitive p^e-th root of unity, where p^e is the group exponent.

Remark 1.6. For convenience of the readers, Table 1 gives a summary for the isoclinism families of nonabelian groups of order p^6 $(p > 3)$ with class ≤ 3.

TABLE 1. Isoclinism families of nonabelian groups with class ≤ 3

Family	Class	$B_0=0$?	Family	Class	$B_0=0$?
Φ_2	2	Yes	Φ_{16}	3	Yes
Φ_3	3	Yes	Φ_{17}	3	Yes
Φ_4	2	Yes	Φ_{18}	3	No
Φ_5	2	Yes	Φ_{19}	3	Yes
Φ_6	3	Yes	Φ_{20}	3	No
Φ_7	3	Yes	Φ_{21}	3	No
Φ_8	3	Yes	Φ_{22}	3	Yes
Φ_{11}	2	Yes	Φ_{31}	3	Yes
Φ_{12}	2	Yes	Φ_{32}	3	Yes
Φ_{13}	2	Yes	Φ_{33}	3	Yes
Φ_{14}	2	Yes	Φ_{34}	3	Yes
Φ_{15}	2	Yes			

The present paper is organized as follows. To prove the vanishing result of Bogomolov multiplier, we first collect some preliminaries on the nonabelian exterior square of a finite group in Section 2. In Section 3, we compute the Bogomolov multiplier case by case for these groups in Table 1 except for $\Phi_{15}, \Phi_{18}, \Phi_{20}$ and Φ_{21}. In Section 4, we answer the Noether's problem for the group $\Phi_{15}(21^4)$ in the family Φ_{15}. As an direct consequence, we conclude that $B_0(\Phi_{15}) = 0$. In the last section, we will use a nonvanishing criterion for Bogomolov multiplier due to Hoshi and Kang [11] to prove $B_0(\Phi_{18}) \neq 0$. The similar arguments can be applied to the situations Φ_{20} and Φ_{21}, the proofs will be omitted.

2. PRELIMINARIES

Let G be a group and $x, y \in G$. We define $x^y = y^{-1}xy$ and write $[x, y] = x^{-1}x^y = x^{-1}y^{-1}xy$ for the commutator of x and y. We define the commutators of higher weight as $[x_1, x_2, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n]$ for $x_1, x_2, \cdots, x_n \in G$. In particular, we write $[x, {}_n y]$ for the commutator $[x_1, y, \cdots, y]$ with n copies of y.

The *nonabelian exterior square* of G, is a group generated by the symbols $x \wedge y$ $(x, y \in G)$, subject to the relations

$$\begin{aligned} xy \wedge z &= (x^y \wedge z^y)(y \wedge z), \\ x \wedge yz &= (x \wedge z)(x^z \wedge y^z), \\ x \wedge x &= 1, \end{aligned}$$

for all $x, y, z \in G$. We denote this group by $G \wedge G$. Let $[G, G]$ be the commutator subgroup of G. We observe that the commutator map $\kappa : G \wedge G \to [G, G]$, given by $x \wedge y \mapsto [x, y]$, is a well-defined group homomorphism. Let $M(G)$ denote the kernel of κ, i.e.,

$$M(G) = \left\{ \prod_{\text{finite}} (x_i \wedge y_i)^{\epsilon_i} \in G \wedge G \mid \epsilon_i = \pm 1, \prod_{\text{finite}} [x_i, y_i]^{\epsilon_i} = 1 \right\}. \tag{2.1}$$

Moreover, we define

$$M_0(G) = \langle x\wedge y\in G\wedge G \mid [x,y]=1\rangle = \left\{\prod_{\text{finite}}(x_i\wedge y_i)^{\epsilon_i}\in G\wedge G \mid \epsilon_i=\pm 1, [x_i,y_i]=1\right\}. \tag{2.2}$$

An important result due to Moravec [20] asserts that $B_0(G)$ is actually isomorphic to $M(G)/M_0(G)$.

Let G be a group. There is also an alternative way to obtain the nonabelian exterior square $G\wedge G$. Let φ be an automorphism of G and G^{φ} be an isomorphic copy of G via $\varphi : x\mapsto x'$. We define $\tau(G)$ to be the group generated by G and G^{φ}, subject to the following relations:

$$[x,y']^z=[x^z,(y^z)']=[x,y']^{z'} \text{ and } [x,x']=1$$

for all $x,y,z\in G$. Obviously, the groups G and G^{φ} can be viewed as subgroups of $\tau(G)$. Let $[G,G^{\varphi}]=\langle [x,y'] \mid x,y\in G\rangle$ be the commutator subgroup. Notice that the map $\phi : G\wedge G\to [G,G^{\varphi}]$ given by $x\wedge y\mapsto [x,y']$ is actually an isomorphism of groups (see [2]).

We collect some properties of $\tau(G)$ and $[G,G^{\varphi}]$ that will be used frequently in our proofs.

Lemma 2.1 ([2])**.** *Let G be a group.*

1. $[x,yz]=[x,z][x,y][x,y,z]$ *and* $[xy,z]=[x,z][x,z,y][y,z]$ *for all* $x,y,z\in G$.
2. *If G is nilpotent of class c, then $\tau(G)$ is nilpotent of class at most $c+1$.*
3. *If G is nilpotent of class ≤ 2, then $[G,G^{\varphi}]$ is abelian.*
4. $[x,y']=[x',y]$ *for all* $x,y\in G$.
5. $[x,y,z']=[x,y',z]=[x',y,z]=[x',y',z]=[x',y,z']=[x,y',z']$ *for all* $x,y,z\in G$.
6. $[[x,y'],[a,b']]=[[x,y],[a,b]']$ *for all* $x,y,a,b\in G$.
7. $[x^n,y']=[x,y']^n=[x,(y')^n]$ *for all integers n and* $x,y\in G$ *with* $[x,y]=1$.
8. *If $[G,G]$ is nilpotent of class c, then $[G,G^{\varphi}]$ is nilpotent of class c or $c+1$.*

Lemma 2.2 ([21], Lemma 3.1)**.** *Let G be a nilpotent group of class ≤ 3. Then*

$$[x,y^n]=[x,y]^n[x,y,y]^{\binom{n}{2}}[x,y,y,y]^{\binom{n}{3}}$$

for all $x,y\in\tau(G)$ and every positive integer n.

We recall several relevant definitions about the polycyclic group. A finite solvable group G is called *polycyclic* if it has a subnormal series $G=G_1\triangleright G_2\triangleright\cdots\triangleright G_{n+1}=1$ such that every factor G_i/G_{i+1} is cyclic of order r_i. A *polycyclic generating sequence* of a finite solvable group G is a sequence $x_1,\cdots,x_n$ of elements of G such that $G_i=\langle G_{i+1},x_i\rangle$ for all $1\leq i\leq n$. The value r_i is called the *relative order* of g_i. Given a polycyclic generating sequence $x_1,\cdots,x_n$, each element x of G can be expressed uniquely as a product $x=x_1^{e_1}\cdots x_n^{e_n}$ with $e_i\in\{0,\cdots,r_i-1\}$. An element x of a polycyclic generating sequence of G is *absolute* if its relative order is equal to the order of x.

Lemma 2.3 ([2], Proposition 20). *Let G be a finite solvable group with a polycyclic generating sequence $x_1, \cdots, x_n$. Then the group $[G, G^\varphi]$ is generated by $\left\{[x_i, x'_j] \mid i, j = 1, \cdots, n, i > j\right\}$.*

In next section, we will follow the route in [21], in which the main idea is the following. Let $\kappa^* = \kappa \cdot \phi^{-1}$ be the composite map from $[G, G^\varphi]$ to $[G, G]$, $M^*(G) = \ker \kappa^*$ and $M_0^*(G) = \phi(M_0(G))$. It is immediate that $B_0(G) \simeq M^*(G)/M_0^*(G)$. Notice that

$$M^*(G) = \left\{\prod_{\text{finite}} [x_i, y'_i]^{\epsilon_i} \in [G, G^\varphi] \mid \epsilon_i = \pm 1, \prod_{\text{finite}} [x_i, y_i]^{\epsilon_i} = 1\right\}, \tag{2.3}$$

and

$$M_0^*(G) = \left\{\prod_{\text{finite}} [x_i, y'_i]^{\epsilon_i} \in [G, G^\varphi] \mid \epsilon_i = \pm 1, [x_i, y_i] = 1\right\}. \tag{2.4}$$

To prove $B_0(G) = 0$, it suffices to show that $M^*(G) \subseteq M_0^*(G)$.

3. Trivial Bogomolov Multipliers

The following theorems are very useful in our discussions.

Theorem 3.1 (Moravec [21]). *Let $p > 3$ be a prime and G be a p-group of class ≤ 3. Let $x_1, \cdots, x_n$ be a polycyclic generating sequence of G. Suppose that all nontrivial commutators $[x_i, x_j] (i > j)$ are different absolute elements of the polycyclic generating sequence. Then $B_0(G) = 0$.*

Theorem 3.2 (Kang [15]). *Let G and H be finite groups. Then $B_0(G \times H)$ is isomorphic to $B_0(G) \times B_0(H)$. As a corollary, if $B_0(G)$ and $B_0(G)$ are both trivial, then also is $B_0(G \times H)$.*

Theorems 3.1, 3.2, 1.1, 1.2 and 1.3 can be applied to groups of order p^6 with class ≤ 3. For example, in the classification of James ([13], page 621), the group $\Phi_2(411)_a = \Phi_2(41) \times (1)$ is the direct product of $\Phi_2(41)$ and a cyclic group of order p. It follows from Theorem 1.2 that $B_0(\Phi_2(41)) = 0$. By Theorem 1.1, we see that $B_0((1)) = 0$. Thus $B_0(\Phi_2(411)_a)$ is trivial by Theorem 3.2. Another example is the group $\Phi_2(51)$, which has a polycyclic presentation $\langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2 = \alpha^{p^4}, \alpha_1^p = \alpha_2^p = 1\rangle$ that satisfies the assumption of Theorem 3.1. Thus $B_0(\Phi_2(51)) = 0$. We continue in this way to check James's classification, it is not hard to see that

Corollary 3.3. *Let $p > 3$ be a prime and G belong to one of the families Φ_i, where*

$$i = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 16, 17, 32.$$

Then $B_0(G) = 0$.

Remark 3.4. Furthermore, Theorem 3.2 can be applied to obtain some groups of class 4 with nontrivial Bogomolov multiplier. For example, the group $\Phi_{10}(1^6) = \Phi_{10}(1^5) \times (1)$. Since $B_0(\Phi_{10}(1^5)) \neq 0$, it follows from Theorem 3.2 that $B_0(\Phi_{10}(1^6)) \neq 0$.

To prove Theorem 1.4, we deal with the remaining cases one by one. In what follows, we always assume that the trivial commutator relations among the generators in a polycyclic presentation of a group will be omitted.

Proposition 3.5. *If* $G \in \Phi_{13}$, *then* $B_0(G) = 0$.

Proof. We choose a representative $G = \Phi_{13}(1^6)$ in the family Φ_{13}. The group G has a polycyclic presentation

$$G = \langle \alpha_1, \cdots, \alpha_4, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta_1, [\alpha_1, \alpha_3] = [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1, (i = 1, 2)\rangle.$$

By Lemma 2.3, the group $[G, G^\varphi]$ is generated modulo $M_0^*(G)$ by $[\alpha_1, \alpha_2']$, $[\alpha_1, \alpha_3']$ and $[\alpha_2, \alpha_4']$. Since $[G, G^\varphi]$ is abelian by (3) of Lemma 2.1, each element in $[G, G^\varphi]$ can be expressed as

$$[\alpha_1, \alpha_2']^r [\alpha_1, \alpha_3']^s [\alpha_2, \alpha_4']^t \widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha_2']^r [\alpha_1, \alpha_3']^s [\alpha_2, \alpha_4']^t \widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta_1^r \beta_2^{s+t}$ and p divides r and $s + t$. Note that $\tau(G)$ is nilpotent of class ≤ 3. So $[\alpha_1', \alpha_2, \alpha_2, \alpha_2] = 1$. We observe that $[\alpha_1', \alpha_2, \alpha_2]^{\binom{p}{2}} = [\alpha_1, \alpha_2, \alpha_2']^{\binom{p}{2}} = [\beta_1, \alpha_2']^{\binom{p}{2}}$. Since $[\beta_1, \alpha_2] = 1$, it follows from (7) of Lemma 2.1 that $[\beta_1, \alpha_2']^{\binom{p}{2}} = [\beta_1^{\binom{p}{2}}, \alpha_2'] = [1, \alpha_2'] = 1$. By Lemma 2.2 we have

$$1 = [\alpha_1', \alpha_2^p] = [\alpha_1', \alpha_2]^p [\alpha_1', \alpha_2, \alpha_2]^{\binom{p}{2}} = [\alpha_1', \alpha_2]^p,$$

i.e., $[\alpha_1, \alpha_2']^p = 1$. Similarly, $[\alpha_1, \alpha_3']^p = 1$ and $[\alpha_2, \alpha_4']^p = 1$, so we have $w = ([\alpha_1, \alpha_3'][\alpha_2, \alpha_4']^{-1})^s \widetilde{w}$. Now, we need to prove that $[\alpha_1, \alpha_3'][\alpha_2, \alpha_4']^{-1} \in M_0^*(G)$. Obverse that

$$\begin{aligned}
[\alpha_1\alpha_2\alpha_4, \alpha_1\alpha_2\alpha_3] &= [\alpha_1, \alpha_3][\alpha_1, \alpha_3, \alpha_2\alpha_4][\alpha_1, \alpha_1\alpha_2][\alpha_1, \alpha_1\alpha_2, \alpha_3\alpha_2\alpha_4][\alpha_2\alpha_4, \alpha_3] \\
&\quad [\alpha_2\alpha_4, \alpha_1\alpha_2][\alpha_2\alpha_4, \alpha_1\alpha_2, \alpha_3] \\
&= [\alpha_1, \alpha_3][\beta_2, \alpha_2\alpha_4][\alpha_1, \alpha_2][\beta_1, \alpha_3\alpha_2\alpha_4][\alpha_2, \alpha_1] \\
&\quad [\alpha_4, \alpha_2][\beta_1^{-1}\beta_2^{-1}, \alpha_3] \\
&= 1.
\end{aligned}$$

Thus $[\alpha_1\alpha_2\alpha_4, (\alpha_1\alpha_2\alpha_3)'] \in M_0^*(G)$. Expanding it, we obtain

$$\begin{aligned}
[\alpha_1\alpha_2\alpha_4, (\alpha_1\alpha_2\alpha_3)'] &= [\alpha_1, \alpha_3'][\alpha_1, \alpha_3', \alpha_2\alpha_4][\alpha_1, \alpha_1'\alpha_2'][\alpha_1, \alpha_1'\alpha_2', \alpha_3'\alpha_2\alpha_4][\alpha_2\alpha_4, \alpha_3'] \\
&\quad [\alpha_2\alpha_4, \alpha_1'\alpha_2'][\alpha_2\alpha_4, \alpha_1'\alpha_2', \alpha_3'] \\
&= [\alpha_1, \alpha_3'][\beta_2, \alpha_2'\alpha_4'][\alpha_1, \alpha_2'][\alpha_1, \alpha_1', \alpha_2'][\alpha_1, \alpha_1'\alpha_2', \alpha_3'\alpha_2\alpha_4][\alpha_2, \alpha_3'][\alpha_2, \alpha_3', \alpha_4] \\
&\quad [\alpha_4, \alpha_3'][\alpha_2, \alpha_1'][\alpha_2, \alpha_1', \alpha_2\alpha_4][\alpha_4, \alpha_2'][\alpha_4, \alpha_1'][\alpha_4, \alpha_1', \alpha_2'][\alpha_2\alpha_4, \alpha_1'\alpha_2', \alpha_3'].
\end{aligned}$$

We can see that, except $[\alpha_1, \alpha_3']$ and $[\alpha_4, \alpha_2']$, the others belong to $M_0^*(G)$. So

$$[\alpha_1, \alpha_3'][\alpha_4, \alpha_2'] = [\alpha_1, \alpha_3'][\alpha_2, \alpha_4']^{-1} \in M_0^*(G),$$

as required. Hence $B_0(G) = 0$. ∎

Proposition 3.6. *If $G \in \Phi_{19}$, then $B_0(G) = 0$.*

Proof. We choose $G = \Phi_{19}(1^6)$ in Φ_{19} as a representative. The group G has a polycyclic presentation

$$G = \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha, \alpha_1] = \beta_1, \alpha_i^p = \alpha^p = \beta^p = \beta_i^p = 1, (i = 1, 2)\rangle.$$

By Lemma 2.3, the group $[G, G^\varphi]$ is generated modulo $M_0^*(G)$ by the set

$$A = \left\{[\alpha_1, \alpha_2'], [\beta, \alpha_1'], [\beta, \alpha_2'], [\alpha, \alpha_1']\right\}.$$

We see that $[\alpha_1, \alpha_2']$ and $[\beta, \alpha_1']$ are commutating elements modulo $M_0^*(G)$. Indeed,

$$\begin{aligned} [[\alpha_1, \alpha_2'], [\beta, \alpha_1']] &= [[\alpha_1, \alpha_2], [\beta, \alpha_1]'] \\ &= [\beta, \beta_1'] \in M_0^*(G). \end{aligned}$$

Similarly, any two elements in the set A are commutating modulo $M_0^*(G)$. Thus, each element in $[G, G^\varphi]$ can be expressed as

$$[\alpha_1, \alpha_2']^m [\beta, \alpha_1']^n [\beta, \alpha_2']^s [\alpha, \alpha_1']^t \widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha_2']^m [\beta, \alpha_1']^n [\beta, \alpha_2']^s [\alpha, \alpha_1']^t \widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta^m \beta_1^{n+t} \beta_2^s$ and p divides m, $n + t$ and s. By (7) of Lemma 2.1 and Lemma 2.2 we have

$$1 = [\alpha_1', \alpha_2^p] = [\alpha_1', \alpha_2]^p [\alpha_1', \alpha_2, \alpha_2]^{\binom{p}{2}} [\alpha_1',_3 \alpha_2]^{\binom{p}{3}} = [\alpha_1', \alpha_2]^p,$$

i.e., $[\alpha_1, \alpha_2']^p = 1$. Similarly, $[\beta, \alpha_i']^p = 1$ and $[\alpha, \alpha_1']^p = 1, (i = 1, 2)$, so we have $w = ([\beta, \alpha_1'][\alpha, \alpha_1']^{-1})^n \widetilde{w}$. Now, we need to prove that $[\beta, \alpha_1'][\alpha, \alpha_1']^{-1} \in M_0^*(G)$. Observe that

$$[\beta\alpha_1, \alpha_1\alpha] = [\beta, \alpha][\beta, \alpha, \alpha_1][\beta, \alpha_1][\beta, \alpha_1, \alpha\alpha_1][\alpha_1, \alpha][\alpha_1, \alpha_1][\alpha_1, \alpha_1, \alpha] = 1.$$

Thus $[\beta\alpha_1, (\alpha_1\alpha)'] \in M_0^*(G)$. Expanding it, we obtain

$$\begin{aligned} [\beta\alpha_1, (\alpha_1\alpha)'] &= [\beta, \alpha'][\beta, \alpha', \alpha_1][\beta, \alpha_1'][\beta, \alpha_1', \alpha'\alpha_1][\alpha_1, \alpha'][\alpha_1, \alpha_1'][\alpha_1, \alpha_1', \alpha'] \\ &= [\beta, \alpha'][\beta, \alpha_1'][\beta, \alpha_1', \alpha_1][\beta, \alpha_1', \alpha][\alpha_1, \alpha']. \end{aligned}$$

We can see that $[\beta, \alpha'], [\beta, \alpha_1', \alpha_1]$, and $[\beta, \alpha_1', \alpha]$ all belong to $M_0^*(G)$. So

$$[\beta, \alpha_1'][\alpha_1, \alpha'] = [\beta, \alpha_1'][\alpha, \alpha_1']^{-1} \in M_0^*(G).$$

Hence $B_0(G) = 0$. □

Proposition 3.7. *If $G \in \Phi_{22}$, then $B_0(G) = 0$.*

Proof. Choose $G = \Phi_{22}(1^6)$ in the family Φ_{22}, which has a polycyclic presentation

$$G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\beta_1, \beta_2] = \alpha_3, \alpha_1^{(p)} = \alpha^p = \alpha_{i+1}^p = \beta_i^p = 1, (i = 1, 2)\rangle,$$

where $\alpha_1^{(p)} = \alpha_1^p \alpha_2^{\binom{p}{2}} \alpha_3^{\binom{p}{3}}$. By Lemma 2.3, the group $[G, G^\varphi]$ is generated modulo $M_0^*(G)$ by $[\alpha_i, \alpha']$, $[\beta_1, \beta_2'], (i = 1, 2)$, among which any two elements are commutating modulo $M_0^*(G)$. It follows that each element in $[G, G^\varphi]$ can be expressed as

$$[\alpha_1, \alpha']^r [\alpha_2, \alpha']^s [\beta_1, \beta_2']^t \widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha']^s[\alpha_2, \alpha']^s[\beta_1, \beta_2']^t\widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \alpha_2^r\alpha_3^{s+t}$ and p divides r and $s+t$. Notice that

$$1 = [\alpha_i', \alpha^p] = [\alpha_i', \alpha]^p[\alpha_i', \alpha, \alpha]^{\binom{p}{2}}[\alpha_i',_3 \alpha]^{\binom{p}{3}} = [\alpha_i', \alpha]^p, (i = 1, 2).$$

Similarly, $[\beta_1, \beta_2']^p = 1$, so we have $w = ([\alpha_2, \alpha'][\beta_1, \beta_2']^{-1})^s\widetilde{w}$. Now, we want to prove that

$$[\alpha_2, \alpha'][\beta_1, \beta_2']^{-1} \in M_0^*(G).$$

Observe that

$$1 = [\alpha_2, \alpha][\beta_2, \beta_1] = [\alpha_2\beta_2, \alpha\beta_1]$$

Thus $[\alpha_2\beta_2, (\alpha\beta_1)'] \in M_0^*(G)$. Since

$$\begin{aligned}
[\alpha_2\beta_2, (\alpha\beta_1)'] &= [\alpha_2, \beta_1'][\alpha_2, \beta_1', \beta_2][\alpha_2, \alpha'][\alpha_2, \alpha', \beta_1'\beta_2][\beta_2, \beta_1'][\beta_2, \alpha'][\beta_2, \alpha', \beta_1'] \\
&= [\alpha_2, \beta_1'][\alpha_2, \alpha'][\alpha_3, \beta_2'][\alpha_3, \beta_1'][\beta_2, \beta_1'][\beta_2, \alpha'],
\end{aligned}$$

and $[\alpha_2, \beta_1'], [\alpha_3, \beta_2'], [\alpha_3, \beta_1'], [\beta_2, \alpha']$ are in $M_0^*(G)$. So $[\alpha_2, \alpha'][\beta_2, \beta_1'] = [\alpha_2, \alpha'][\beta_1, \beta_2']^{-1} \in M_0^*(G)$. Hence $B_0(G) = 0$. □

Proposition 3.8. *If $G \in \Phi_{31}$, then $B_0(G) = 0$.*

Proof. Choose $G = \Phi_{31}(1^6)$ in the family Φ_{31}, which has a polycyclic presentation

$$G = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \mid [\alpha_i, \alpha] = \beta_i, [\alpha_i, \beta_i] = \gamma, \alpha_i^p = \alpha^p = \beta_i^p = \gamma^p = 1, (i = 1, 2)\rangle.$$

By Lemma 2.3, the group $[G, G^\varphi]$ is generated modulo $M_0^*(G)$ by $[\alpha_i, \alpha']$, $[\alpha_i, \beta_i'], (i = 1, 2)$. Any two elements among these generators are commutating modulo $M_0^*(G)$. Thus each element in $[G, G^\varphi]$ can be expressed as

$$[\alpha_1, \alpha']^m[\alpha_2, \alpha']^n[\alpha_1, \beta_1']^s[\alpha_2, \beta_2']^t\widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha']^m[\alpha_2, \alpha']^n[\alpha_1, \beta_1']^s[\alpha_2, \beta_2']^t\widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta_1^m\beta_2^n\gamma^{s+t}$ and p divides m, n and $s+t$. By Lemma 2.1 and Lemma 2.2 we have

$$1 = [\alpha_i', \alpha^p] = [\alpha_i', \alpha]^p[\alpha_i', \alpha, \alpha]^{\binom{p}{2}}[\alpha_i',_3 \alpha]^{\binom{p}{3}} = [\alpha_i', \alpha]^p, (i = 1, 2).$$

Similarly, $[\alpha_i, \beta_i']^p = 1, (i = 1, 2)$, so we have $w = ([\alpha_1, \beta_1'][\alpha_2, \beta_2']^{-1})^s\widetilde{w}$. To complete the proof, it suffices to prove that $[\alpha_1, \beta_1'][\alpha_2, \beta_2']^{-1} \in M_0^*(G)$. Observe that

$$\begin{aligned}
[\alpha_1\beta_2, \beta_1\alpha_2] &= [\alpha_1, \alpha_2][\alpha_1, \alpha_2, \beta_2][\alpha_1, \beta_1][\alpha_1, \beta_1, \alpha_2\beta_2][\beta_2, \alpha_2][\beta_2, \beta_1][\beta_2, \beta_1, \alpha_2] \\
&= [\alpha_1, \beta_1][\gamma, \alpha_2\beta_2][\beta_2, \alpha_2] \\
&= 1.
\end{aligned}$$

Thus $[\alpha_2\beta_2, (\beta_1\alpha_2)'] \in M_0^*(G)$. Expanding it, we obtain

$$\begin{aligned}
[\alpha_1\beta_2, (\beta_1\alpha_2)'] &= [\alpha_1, \alpha_2'][\alpha_1, \alpha_2', \beta_2][\alpha_1, \beta_1'][\alpha_1, \beta_1', \alpha_2'\beta_2][\beta_2, \alpha_2'][\beta_2, \beta_1'][\beta_2, \beta_1', \alpha_2'] \\
&= [\alpha_1, \alpha_2'][\alpha_1, \beta_1'][\gamma, \beta_2'][\gamma, \alpha_2'][\beta_2, \alpha_2'][\beta_2, \beta_1'].
\end{aligned}$$

Notice that $[\alpha_1,\alpha_2'],[\gamma,\beta_2'],[\gamma,\alpha_2']$, and $[\beta_2,\beta_1']$, belong to $M_0^*(G)$. Therefore

$$[\alpha_1,\beta_1'][\beta_2,\alpha_2'] = [\alpha_1,\beta_1'][\alpha_2,\beta_2']^{-1} \in M_0^*(G),$$

as desired. □

Proposition 3.9. *If $G \in \Phi_{33}$, then $B_0(G) = 0$.*

Proof. Let $G = \Phi_{33}(1^6)$ be a group in the family Φ_{33}. The group G has a polycyclic presentation

$G = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \mid [\alpha_i,\alpha] = \beta_i, [\alpha_1,\beta_1] = [\beta_2,\alpha] = \gamma, \alpha^p = \alpha_1^p = \alpha_2^{(p)} = \beta_i^p = \gamma^p = 1, (i = 1,2)\rangle$,

where $\alpha_2^{(p)} = \alpha_2^p\gamma^{\binom{p}{3}}$. Notice that the group $[G,G']$ is generated modulo $M_0^*(G)$ by $[\alpha_i,\alpha'],(i = 1,2)$, $[\alpha_1,\beta_1']$, and $[\beta_2,\alpha']$, and every element in $[G,G^\varphi]$ can be expressed as

$$[\alpha_1,\alpha']^m[\alpha_2,\alpha']^n[\alpha_1,\beta_1']^s[\beta_2,\alpha']^t\widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1,\alpha']^m[\alpha_2,\alpha']^n[\alpha_1,\beta_1']^s[\beta_2,\alpha']^t\widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta_1^m\beta_2^n\gamma^{s+t}$ and p divides m, n and $s+t$. Notice that

$$1 = [\alpha_i',\alpha^p] = [\alpha_i',\alpha]^p[\alpha_i',\alpha,\alpha]^{\binom{p}{2}}[\alpha_i',_3\alpha]^{\binom{p}{3}} = [\alpha_i',\alpha]^p, (i = 1,2).$$

and similarly, $[\alpha_1,\beta_1']^p = 1, [\beta_2,\alpha']^p = 1$. Thus we have $w = ([\alpha_1,\beta_1'][\beta_2,\alpha']^{-1})^s\widetilde{w}$. Observe that

$$[\alpha_1\alpha,\beta_1\beta_2] = [\alpha_1,\beta_2][\alpha_1,\beta_2,\alpha][\alpha_1,\beta_1][\alpha_1,\beta_1,\beta_2\alpha][\alpha,\beta_2][\alpha,\beta_1][\alpha,\beta_1,\beta_2] = 1$$

Thus $[\alpha_1\alpha,\beta_1\beta_2] \in M_0^*(G)$, and on the other hand,

$$\begin{aligned}[\alpha_1\alpha,(\beta_1\beta_2)'] &= [\alpha_1,\beta_2'][\alpha_1,\beta_2',\alpha][\alpha_1,\beta_1'][\alpha_1,\beta_1',\beta_2'\alpha][\alpha,\beta_2'][\alpha,\beta_1'][\alpha,\beta_1',\beta_2'] \\ &= [\alpha_1,\beta_2'][\alpha_1,\beta_1'][\gamma,\alpha'][\gamma,\beta_2'][\alpha,\beta_2'][\alpha,\beta_1'].\end{aligned}$$

Since $[\alpha_1,\beta_2'],[\gamma,\alpha'],[\gamma,\beta_2']$, and $[\alpha,\beta_1']$ are in $M_0^*(G)$,

$$[\alpha_1,\beta_1'][\alpha,\beta_2'] = [\alpha_1,\beta_1'][\beta_2,\alpha']^{-1} \in M_0^*(G),$$

Thus $B_0(G) = 0$. □

Proposition 3.10. *If $G \in \Phi_{34}$, then $B_0(G) = 0$.*

Proof. We choose $G = \Phi_{34}(321)_a$ as a representative. The group G has a polycyclic presentation

$$\begin{aligned}G = \langle \alpha,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma \quad | \quad & [\alpha_i,\alpha] = \beta_i, [\beta_2,\alpha] = [\alpha_1,\beta_1] = \beta_1^p = \gamma, \\ & \alpha^p = \beta_1, \alpha_1^p = \beta_2, \alpha_2^p = \beta_2^p = \gamma^p = 1 (i = 1,2)\rangle.\end{aligned}$$

We notice that the group $[G,G^\varphi]$ is generated modulo $M_0^*(G)$ by $[\alpha_1,\alpha']$, $[\alpha_2,\alpha']$, $[\beta_2,\alpha']$, and $[\alpha_1,\beta_1']$. It is easy to see that each element in $[G,G^\varphi]$ can be expressed as

$$[\alpha_1,\alpha']^m[\alpha_2,\alpha']^n[\beta_2,\alpha']^s[\alpha_1,\beta_1']^t\widetilde{w},$$

where $\widetilde{w} \in M_0^*(G)$. Let $w = [\alpha_1, \alpha']^m[\alpha_2, \alpha']^n[\beta_2, \alpha']^s[\alpha_1, \beta_1']^t\widetilde{w} \in M^*(G)$. Then $1 = \kappa^*(w) = \beta_1^m\beta_2^n\gamma^{s+t}$ and p divides n, $s+t$, and p^2 divides m. Note that

$$1 = [\alpha', \beta_2^p] = [\alpha', \beta_2]^p[\alpha', \beta_2, \beta_2]^{\binom{p}{2}}[\alpha',_3 \beta_2]^{\binom{p}{3}} = [\alpha', \beta_2]^p.$$

Similarly, $[\alpha_1, \alpha']^{p^2} = 1$, $[\alpha_2, \alpha']^p = 1$ and $[\alpha_1, \beta_1']^p \in M_0^*(G)$, so we have $w = ([\beta_2, \alpha'][\alpha_1, \beta_1]^{-1})^t\widetilde{w}$. We observe that

$$[\beta_2\beta_1, \alpha\alpha_1] = [\beta_2, \alpha_1][\beta_2, \alpha_1, \beta_1][\beta_2, \alpha][\beta_2, \alpha, \alpha_1\beta_1][\beta_1, \alpha_1][\beta_1, \alpha][\beta_1, \alpha, \alpha_1] = 1.$$

Thus $[\beta_2\beta_1, (\alpha\alpha_1)'] \in M_0^*(G)$. Expanding it, we obtain

$$\begin{aligned} [\beta_2\beta_1, (\alpha\alpha_1)'] &= [\beta_2, \alpha_1'][\beta_2, \alpha_1', \beta_1][\beta_2, \alpha'][\beta_2, \alpha', \alpha_1'\beta_1][\beta_1, \alpha_1'][\beta_1, \alpha'][\beta_1, \alpha', \alpha_1'] \\ &= [\beta_2, \alpha_1'][\beta_2, \alpha'][\gamma, \beta_1'][\gamma, \alpha_1'][\beta_1, \alpha_1'][\beta_1, \alpha']. \end{aligned}$$

Since $[\beta_2, \alpha_1'], [\gamma, \beta_1'], [\gamma, \alpha_1'], [\beta_1, \alpha']$ are all in $M_0^*(G)$,

$$[\beta_2, \alpha'][\beta_1, \alpha_1'] = [\beta_2, \alpha'][\alpha_1, \beta_1']^{-1} \in M_0^*(G).$$

Hence $B_0(G) = 0$ and we are done. □

4. Noether's Problem For Φ_{15}

Let k be any field and G be a finite group acting on the rational function field $k(x_h : h \in G)$ by $g \cdot x_h = x_{gh}$ for all $g, h \in G$. We write $k(G)$ for the fixed field $k(x_h : h \in G)^G$. The main purpose of this section is to prove that if $G = \Phi_{15}(21^4)$ and k contains a primitive p^2-th root of unity, then $k(G)$ is rational over k. As a direct consequence, we have $B_0(G) = 0$. To do this, we need some well-known results which will be used in our proof.

Lemma 4.1 (No-name Lemma ([14], page 22)). *Let G be a finite group acting faithfully on a finite-dimensional k-vector space V, and let W be a faithful $k[G]$-submodule of V. Then the extension of the fixed fields $k(V)^G/k(W)^G$ is rational.*

Theorem 4.2 ([1]). *Let L be a field and G be a finite group acting on the rational function field $L(x)$. Assume that for any $g \in G$, $g(L) \subseteq L$ and $g(x) = a_g \cdot x + b_g$, where $a_g, b_g \in L$ and $a_g \neq 0$. Then $L(x)^G = L^G(f)$ for some polynomial $f \in L[x]$.*

Theorem 4.3 ([10]). *Let p be a prime and G a p-group of order $\leq p^4$. Assume that k is a field of char$k \neq p$ and contains a primitive p^eth root of unity, where p^e is the exponent of G. Then $k(V)^G$ is purely transcendental over k for any linear representation V.*

Recall that a k-automorphism $\beta \in \mathrm{Aut}_k k(x_1, \cdots, x_m)$ is said to be *linearized* if there exists an injection from the cyclic group $\langle\beta\rangle$ to $GL_m(k)$. Equivalently, β is linearized if and only if there are m elements $z_1, \cdots, z_m \in k(x_1, \cdots, x_m)$ such that $k(x_1, \cdots, x_m) = k(z_1, \cdots, z_m)$ and $\beta \cdot (z_i) = \sum_{j=1}^m b_{ij}z_j$, where (b_{ij}) is an $m \times m$ invertible matrix over k.

Lemma 4.4 ([7], Corollary 2.2)**.** *Let $m = p^s - 1$ and β be a k-automorphism of $k(x_1, \cdots, x_m)$ with the action*

$$\begin{aligned} \beta \quad : \quad & x_1 \mapsto x_1 x_2^{p^t} \\ & x_2 \mapsto x_3 \mapsto \cdots \mapsto x_{p^s-1} \mapsto \frac{1}{x_1^{p^{s-t}} x_2^{p^s-1} x_3^{p^s-2} \cdots x_{p^s-1}^2} \mapsto x_1^{p^{s-t}} x_2^{p^s-2} x_3^{p^s-3} \cdots x_{p^s-2}^2 x_{p^s-1} \mapsto x_2, \end{aligned}$$

where $s \geq t$. Then β is linearized.

The following is our main result of this section.

Theorem 4.5. *Let $G = \Phi_{15}(21^4) = \langle \alpha_1, \cdots, \alpha_4, \beta_1, \beta_2 \mid \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta_1^p = \beta_2^p = 1, [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1 = \alpha_1^p, [\alpha_1, \alpha_3] = \beta_2, [\alpha_2, \alpha_4] = \beta_2^\theta \rangle$ be a nonabelian group of order p^6, where θ is the smallest positive integer which is a primitive root (mod p). Assume that the base field k contains a primitive p^2-th root of unity. Then the fixed field $k(G)$ is rational over k. In particular, $B_0(G) = 0$.*

Proof. Let η be a primitive p^2-th root of unity. Then $\omega = \eta^p$ is a primitive p-th root of unity. Let $V = \oplus_{g \in G} k \cdot x_g$ be the regular representation of G. Define

$$X_1 = \sum_{0 \leq j \leq p^2-1} x_{\alpha_1^j}, \quad X_2 = \sum_{0 \leq j \leq p-1} x_{\alpha_4^j}.$$

Then $\alpha_1 \cdot X_1 = X_1$ and $\alpha_4 \cdot X_2 = X_2$. Define

$$\begin{aligned} Y_1 &= \sum_{0 \leq j \leq p-1} \omega^{-j} \alpha_4^j \cdot X_1 = X_1 + \omega^{-1} \alpha_4 \cdot X_1 + \cdots + \omega^{-(p-1)} \alpha_4^{p-1} \cdot X_1. \\ Y_2 &= \sum_{0 \leq j \leq p^2-1} \eta^{-j} \alpha_1^j \cdot X_2 = X_2 + \eta^{-1} \alpha_1 \cdot X_2 + \cdots + \eta^{-(p^2-1)} \alpha_1^{p^2-1} \cdot X_2. \end{aligned}$$

Since $[\alpha_1, \alpha_4] = 1$, it follows that

$$\begin{aligned} \alpha_1 \quad &: \quad Y_1 \mapsto Y_1, \quad Y_2 \mapsto \eta \cdot Y_2 \\ \alpha_4 \quad &: \quad Y_1 \mapsto \omega \cdot Y_1, \quad Y_2 \mapsto Y_2. \end{aligned}$$

Notice that $\beta_1 = \alpha_1^p$. We define

$$\widetilde{Y_1} = \sum_{0 \leq j \leq p-1} \beta_1^j \cdot Y_1, \quad \widetilde{Y_2} = \sum_{0 \leq j \leq p-1} \omega^{-j} \beta_1^j \cdot Y_2.$$

Since β_1 belongs to the center of G, we have

$$\begin{aligned} \alpha_1 \quad &: \quad \widetilde{Y_1} \mapsto \widetilde{Y_1}, \quad \widetilde{Y_2} \mapsto \eta \cdot \widetilde{Y_2} \\ \alpha_4 \quad &: \quad \widetilde{Y_1} \mapsto \omega \cdot \widetilde{Y_1}, \quad \widetilde{Y_2} \mapsto \widetilde{Y_2} \\ \beta_1 \quad &: \quad \widetilde{Y_1} \mapsto \widetilde{Y_1}, \quad \widetilde{Y_2} \mapsto \omega \cdot \widetilde{Y_2}. \end{aligned}$$

Define

$$\widetilde{X_1} = \sum_{0 \leq j \leq p-1} \omega^{-j} \beta_2^j \cdot \widetilde{Y_1}, \quad \widetilde{X_2} = \sum_{0 \leq j \leq p-1} \omega^{-j} \beta_2^j \cdot \widetilde{Y_2}.$$

Since β_2 is also an element in the center of G, it follows that

$$\begin{aligned}
\alpha_1 &: \widetilde{X_1} \mapsto \widetilde{X_1}, \quad \widetilde{X_2} \mapsto \eta \cdot \widetilde{X_2} \\
\alpha_4 &: \widetilde{X_1} \mapsto \omega \cdot \widetilde{X_1}, \quad \widetilde{X_2} \mapsto \widetilde{X_2} \\
\beta_1 &: \widetilde{X_1} \mapsto \widetilde{X_1}, \quad \widetilde{X_2} \mapsto \omega \cdot \widetilde{X_2} \\
\beta_2 &: \widetilde{X_1} \mapsto \omega \cdot \widetilde{X_1}, \quad \widetilde{X_2} \mapsto \omega \cdot \widetilde{X_2}.
\end{aligned}$$

For $0 \leq i \leq p-1$, we define

$$\begin{aligned}
x_i &= \alpha_2^i \cdot \left(\sum_{j,k=0}^{p-1} \omega^k \alpha_3^j \cdot \widetilde{X_1} \right) \\
y_i &= \alpha_3^i \cdot \left(\sum_{j,k=0}^{p-1} \omega^k \alpha_2^j \cdot \widetilde{X_2} \right).
\end{aligned}$$

We observe that

$$\begin{aligned}
\alpha_1 &: x_i \mapsto x_i, \quad y_i \mapsto \eta\omega^i \cdot y_i \\
\alpha_2 &: x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{p-1} \mapsto x_0, \quad y_i \mapsto y_i \\
\alpha_3 &: x_i \mapsto x_i, \quad y_0 \mapsto y_1 \mapsto \cdots \mapsto y_{p-1} \mapsto y_0 \\
\alpha_4 &: x_i \mapsto \omega^{-\theta i+1} \cdot x_i, \quad y_i \mapsto \omega^{-i} \cdot y_i \\
\beta_1 &: x_i \mapsto x_i, \quad y_i \mapsto \omega \cdot y_i \\
\beta_2 &: x_i \mapsto \omega \cdot x_i, \quad y_i \mapsto \omega \cdot y_i.
\end{aligned}$$

We see that $W = (\oplus_{0\leq i\leq p-1} k \cdot x_i) \oplus (\oplus_{0\leq i\leq p-1} k \cdot y_i)$ is a faithful G-subrepresentation of V. By No-name Lemma 4.1, it suffices to show that the invariant field $k(W)^G$ is rational over k.

For $1 \leq i \leq p-1$, we define $u_i = \frac{x_i}{x_{i-1}}$ and $v_i = \frac{y_i}{y_{i-1}}$. Then $k(W) = k(x_0, y_0, u_i, v_i : 1 \leq i \leq p-1)$. Let $L = k(u_i, v_i : 1 \leq i \leq p-1)$. For any $g \in G$, we have that $g \cdot x_0 = \ell_g \cdot x_0$ and $g \cdot y_0 = \widetilde{\ell_g} \cdot y_0$, where $\ell_g, \widetilde{\ell_g} \in L$. Notice that L is invariant by the action of G. More precisely,

$$\begin{aligned}
\alpha_1 &: u_i \mapsto u_i, \quad v_i \mapsto \omega \cdot v_i \\
\alpha_2 &: u_1 \mapsto u_2 \mapsto \cdots \mapsto u_{p-1} \mapsto (u_1 u_2 \cdots u_{p-1})^{-1}, \quad v_i \mapsto v_i \\
\alpha_3 &: u_i \mapsto u_i, \quad v_1 \mapsto v_2 \mapsto \cdots \mapsto v_{p-1} \mapsto (v_1 v_2 \cdots v_{p-1})^{-1} \\
\alpha_4 &: u_i \mapsto \omega^{-\theta} \cdot u_i, \quad v_i \mapsto \omega^{-1} \cdot v_i \\
\beta_1, \beta_2 &: u_i \mapsto u_i, \quad v_i \mapsto v_i.
\end{aligned}$$

By Theorem 4.2, we need only to prove that L^G is rational over k. Obviously, β_1 and β_2 act trivially on L, so $L^G = L^{\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle}$.

It follows that $L^{\langle\alpha_1,\alpha_4\rangle} = k(u_i, v_i : 1 \le i \le p-1)^{\langle\alpha_1,\alpha_4\rangle} = k(r_i, z_i : 1 \le i \le p-1)$, where $r_1 = u_1^p, z_1 = v_1^p$ and $r_i = \frac{u_i}{u_{i-1}}, z_i = \frac{v_i}{v_{i-1}}$ for $2 \le i \le p-1$. The actions of α_2 and α_3 on $k(r_i, z_i : 1 \le i \le p-1)$ are given by

$$\begin{aligned}
\alpha_2 \ :\ & r_1 \mapsto r_1 r_2^p, \\
& r_2 \mapsto r_3 \mapsto \cdots \mapsto r_{p-1} \mapsto \frac{1}{r_1 r_2^{p-1} r_3^{p-2} \cdots r_{p-1}^2} \mapsto r_1 r_2^{p-2} r_3^{p-3} \cdots r_{p-2}^2 r_{p-1} \mapsto r_2; \\
& z_i \mapsto z_i. \\
\alpha_3 \ :\ & z_1 \mapsto z_1 z_2^p, \\
& z_2 \mapsto z_3 \mapsto \cdots \mapsto z_{p-1} \mapsto \frac{1}{z_1 z_2^{p-1} z_3^{p-2} \cdots z_{p-1}^2} \mapsto z_1 z_2^{p-2} z_3^{p-3} \cdots z_{p-2}^2 z_{p-1} \mapsto z_2; \\
& r_i \mapsto r_i.
\end{aligned}$$

It follows from Lemma 4.4 that the actions of α_2, α_3 on $k(r_i, z_i : 1 \le i \le p-1)$ can be linearized simultaneously. Applying Theorem 4.3, we conclude that $k(r_i, z_i : 1 \le i \le p-1)^{\langle\alpha_2,\alpha_3\rangle}$ is rational over k. This completes the proof. □

Remark 4.6. The method above can be applied to discuss the Noether's problem for other groups in the family Φ_{15}.

5. Nontrivial Bogomolov Multipliers

In this section, we use the following nonvanishing criterion for the Bogomolov multiplier to complete the proof of Theorem 1.4.

Lemma 5.1 (Hoshi-Kang [11])**.** *Let G be a finite group, N be a normal subgroup of G. Assume that*

(1) *the transgression map tr: $H^1(N, \mathbf{Q}/\mathbf{Z})^G \to H^2(G/N, \mathbf{Q}/\mathbf{Z})$ is not surjective, and*

(2) *for any bicyclic subgroup A of G, the group AN/N is a cyclic subgroup of G/N.*

Then $B_0(G) \neq 0$.

Proposition 5.2. *If $G \in \Phi_{18}$, then $B_0(G) \neq 0$.*

Proof. We choose $G = \Phi_{18}(1^6)$ as a representative. The group G has a polycyclic presentation

$$\begin{aligned}
G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \ \mid\ & [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, [\alpha, \beta] = \gamma, \\
& \alpha^p = \beta^p = \alpha_1^p = \alpha_{i+1}^p = \gamma^p = 1 (i = 1, 2)\rangle.
\end{aligned}$$

Let $N = \langle \alpha_3, \beta, \gamma \rangle$ be the normal subgroup of G. We will prove that N satisfies the two conditions in Lemma 5.1, thus $B_0(G) \neq 0$.

Since $N \cong C_p \times C_p \times C_p$, it follows that $H^1(N, \mathbf{Q}/\mathbf{Z}) = \mathrm{Hom}(N, \mathbf{Q}/\mathbf{Z}) \cong C_p \times C_p \times C_p$. Define $\varphi_1, \varphi_2, \varphi_3 \in H^1(N, \mathbf{Q}/\mathbf{Z})$ by

$$\varphi_1(\alpha_3) = \frac{1}{p}, \varphi_1(\beta) = 0, \varphi_1(\gamma) = 0;$$

$$\varphi_2(\alpha_3) = 0, \varphi_2(\beta) = \frac{1}{p}, \varphi_2(\gamma) = 0;$$
$$\varphi_3(\alpha_3) = 0, \varphi_3(\beta) = 0, \varphi_3(\gamma) = \frac{1}{p}.$$

We have $H^1(N, \mathbf{Q}/\mathbf{Z}) = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$. The action of G on $\varphi_1, \varphi_2, \varphi_3$ are given by

$$\alpha \cdot \varphi_1(\alpha_3) = \varphi_1(\alpha^{-1}\alpha_3\alpha) = \varphi_1(\alpha_3) = \frac{1}{p};$$
$$\alpha \cdot \varphi_1(\beta) = \varphi_1(\alpha^{-1}\beta\alpha) = \varphi_1(\beta\gamma^{-1}) = \varphi_1(\beta) + \varphi_1(\gamma^{-1}) = 0;$$
$$\alpha \cdot \varphi_1(\gamma) = \varphi_1(\alpha^{-1}\gamma\alpha) = \varphi_1(\gamma) = 0.$$

Thus α fixes φ_1. Similarly,

$$\alpha \cdot \varphi_2(\alpha_3) = 0, \alpha \cdot \varphi_2(\beta) = \frac{1}{p}, \alpha \cdot \varphi_2(\gamma) = 0,$$
$$\alpha \cdot \varphi_3(\alpha_3) = 0, \alpha \cdot \varphi_3(\beta) = -\frac{1}{p}, \alpha \cdot \varphi_3(\gamma) = \frac{1}{p}.$$

Hence, $\alpha \cdot \varphi_3 = -\varphi_2 + \varphi_3$ and α fixes φ_2. With an analogous argument, we eventually obtain

$$\alpha_1 : \varphi_1 \mapsto \varphi_1 - \varphi_2, \varphi_2 \mapsto \varphi_2, \varphi_3 \mapsto \varphi_3;$$
$$\alpha_2 : \varphi_1 \mapsto \varphi_1, \varphi_2 \mapsto \varphi_2, \varphi_3 \mapsto \varphi_3.$$

For any $\varphi \in H^1(N, \mathbf{Q}/\mathbf{Z})$, we write $\varphi = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$ for some integers $a_1, a_2, a_3 \in \mathbf{Z}$ (modulo p). It is easy to check that $\varphi \in H^1(N, \mathbf{Q}/\mathbf{Z})^G$ if and only if $a_1 = a_3 = 0$. Obviously, $\varphi_2 \in H^1(N, \mathbf{Q}/\mathbf{Z})^G$. Thus $H^1(N, \mathbf{Q}/\mathbf{Z})^G = \langle \varphi_2 \rangle \cong C_p$. Notice that G/N is a nonabelian group of order p^3 and of exponent p, it follows from Proposition 6.3 in [18] (or see [16], Theorem 3.3.6) that $H^2(G/N, \mathbf{Q}/\mathbf{Z}) \cong C_p \times C_p$. Therefore, the transgression map tr: $H^1(N, \mathbf{Q}/\mathbf{Z})^G \to H^2(G/N, \mathbf{Q}/\mathbf{Z})$ is not surjective.

The second step is to prove that the group AN/N is a cyclic subgroup of G/N for any bicyclic subgroup A of G. Recall that a group A is said to be *bicyclic* if A is either cyclic or a direct product of two cyclic groups. The following formulae follows from the commutator relations of G:

$$\alpha_2^i\alpha^j = \alpha^j\alpha_2^i\alpha_3^{ij}, \tag{5.1}$$
$$\alpha_1^i\beta^j = \beta^j\alpha_1^i\alpha_3^{ij}, \tag{5.2}$$
$$\alpha_1^i\alpha^j = \alpha^j\alpha_1^i\alpha_2^{ij}\alpha_3^{i\cdot\binom{j}{2}}, \tag{5.3}$$
$$\alpha^i\beta^j = \beta^j\alpha^i\gamma^{ij}, \tag{5.4}$$

where $1 \leq i, j \leq p-1$ and $\binom{x}{y}$ denotes the binomial coefficient when $x \geq y \geq 1$ and we adopt the convention $\binom{x}{y} = 0$ if $1 \leq x < y$. Let $A = \langle y_1, y_2 \rangle$ be a bicyclic subgroup of G. We observe that AN/N is abelian and G/N is nonabelian, so AN/N is a proper subgroup of G/N. Thus the order of AN/N is either p or p^2. If the order of AN/N is p, then it is cyclic, we done.

Assume that the order of AN/N is p^2, we will prove that this is impossible. In G/N, we write $y_1N = \alpha^{a_1}\alpha_1^{a_2}\alpha_2^{a_3}N$ and $y_2N = \alpha^{b_1}\alpha_1^{b_2}\alpha_2^{b_3}N$ for some integers a_i, b_i. The almost same proof as in Lemma 2.2 of

Hoshi-Kang [11] implies that there are only three possibilities:

$$(y_1N, y_2N) = (\alpha_1 N, \alpha_2 N), (\alpha\alpha_2^{a_3}N, \alpha_1\alpha_2^{b_3}N), (\alpha\alpha_1^{a_2}N, \alpha_2 N)$$

if it is necessary to change some suitable generators y_1, y_2 and integers a_2, a_3, b_3. Finally we will show that all three possibilities lead to contradiction. For the first case, we write $y_1 = \alpha_1\alpha_3^{a_4}\beta^{a_5}\gamma^{a_6}$ and $y_2 = \alpha_2\alpha_3^{b_4}\beta^{b_5}\gamma^{b_6}$ for for some integers a_j, b_j. Since y_1 and y_2 are commutating, $\alpha_1\beta^{a_5}\alpha_2\beta^{b_5} = \alpha_2\beta^{b_5}\alpha_1\beta^{a_5}$. It follows from (5.2) that $[\alpha_1, \alpha_2] \neq 1$, which is a contradiction. Second, suppose $y_1N = \alpha\alpha_2^{a_3}N$ and $y_2N = \alpha_1\alpha_2^{b_3}N$. Using (5.1) and (5.3), we obtain that y_1N and y_2N do not commutate. This is a contradiction again. The last case is similar. We write $y_1 = \alpha\alpha_1^{a_2}\alpha_3^{a_4}\beta^{a_5}\gamma^{a_6}$ and $y_2 = \alpha_2\alpha_3^{b_4}\beta^{b_5}\gamma^{b_6}$. Notice that $y_1y_2 = y_2y_1$, but we use (5.1)-(5.4) to get a contradictory fact that y_1 and y_2 do not commutate.

The proof is completed. □

One can apply the same techniques to the case Φ_{21} and Φ_{20}. For instance, we choose $\Phi_{21}(1^6)$ as a representative. It has a polycyclic presentation

$$\langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \mid [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha, \alpha_1] = \beta_2, [\alpha, \alpha_2] = \beta_1^{\nu}, \\ \alpha^p = \alpha_i^p = \beta^p = \beta_i^p = 1 (i = 1, 2)\rangle,$$

where ν is the smallest positive integer which is a non-quadratic residue (mod p). To prove the group $\Phi_{21}(1^6)$ satisfies the two conditions in Lemma 5.1, the most key step is to choose the suitable normal subgroup N. Here we choose $N = \langle \beta, \beta_1, \beta_2\rangle$ and it is routine to check these conditions as in the proof of Proposition 5.2. We eventually obtain the following result (the detailed proof is omitted).

Proposition 5.3. *If $G \in \Phi_{21}$ or Φ_{20}, then $B_0(G) \neq 0$.*

Acknowledgments

The first-named author would like to thank Professors Hang Gao, Ming-chang Kang and Primož Moravec for their help.

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